

domain decomposed implicit methods for model magnetohydrodynamics problems

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magnetohydrodynamics (mhd)

- a mathematical model that describes the motion of a continuous, electrically conducting fluid (plasma), in the presence of a magnetic field.
- the physics is described by: the navier-stokes equations, the maxwell equations and ohm's law.
- the phenomena are characterized by a multiphysics, multirate, multiscale nature.
- mhd models: ideal (one fluid), resistive (one fluid), hall (two fluids).
- case studies: magnetic reconnection, tilt mode instability.

motivation

- there is a need for robust, fast and accurate solvers for simulating such challenging phenomena on very fine grids using a very large number of processors.
- traditional approaches use explicit methods, operator splitting.
- for stability, explicit methods must resolve the fastest phenomena and are therefore subject to the courant-friedrichs-lewy (cfl) condition ($\Delta t_{explicit} \leq \Delta t_{CFL}$).
- stability can be achieved with time steps that are orders of magnitude larger, if an implicit method is used ($\Delta t_{implicit} \gg \Delta t_{CFL}$).
- a fully implicit solution is more accurate: it lacks the operator splitting errors and allows high order time integration.
- a fully implicit solution benefits from the robustness of legacy solvers.

computational platform and parameters

- **ibm bg/l (frost) at ncar: 1024 nodes, 2 processors (700 mhz) per node, 512 mbyte memory per node.**
- **petsc 2.3.1: parallel data structures, parallel linear (schwarz preconditioners and krylov accelerators) and nonlinear (newton) solvers.**
- **rectangular grids from 100×100 to 2000×2000 .**
- **sparse linear systems with coefficient matrices sizes from 160,000 to 64,000,000 and with 3,200,000 to 1,280,000,000 nonzeros.**
- **memory requirements from 25,600,000 to 10,240,000,000 bytes.**

case study: magnetic reconnection (1/3)

- collaboration with amitava bhattacharjee and kai germanchewski from university of new hampshire.
- $[0, 2\pi] \times [0, 4\pi]$ domain with double periodic boundary conditions.
- unknowns: Ω (vorticity), J (current density), F (average canonical momentum), Φ (vorticity stream function), Ψ (current density stream function).
- parameters: μ (viscosity), η (resistivity), d_e (inertial skin depth), ρ_s (ion sound larmor radius).

case study: magnetic reconnection (2/3)

- **steady state (equilibrium):** $\Omega^e = 0$, $J^e = \cos(x)$, $F^e = (1 + d_e^2) \cos(x)$, $\Phi^e = 0$, $\Psi^e = \cos(x)$.
- **perturbation Φ^p applied to the steady state.**
- **after the perturbation is applied the four variables have both equilibrium and perturbed components:** $\Omega = \Omega^e + \Omega^p$, $J = J^e + J^p$, $F = F^e + F^p$, $\Phi = \Phi^e + \Phi^p$, $\Psi = \Psi^e + \Psi^p$.
- **study the evolution of the perturbed components:** Ω^p , J^p , F^p , Φ^p , Ψ^p .
- **note:** $[f, g] = \frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial g}{\partial x}$

case study: magnetic reconnection (3/3)

$$\frac{\partial \Omega}{\partial t} + [\Phi, \Omega] = d_e^{-2} [F, \Psi] + \mu \nabla^2 \Omega$$

$$\frac{\partial F}{\partial t} + [\Phi, F] = \rho_s^2 [\Omega, \Psi] + \eta \nabla^2 (\Psi - \Psi^e)$$

$$\nabla^2 \Phi = \Omega$$

$$\nabla^2 \Psi = d_e^{-2} (\Psi - F)$$

$$J = d_e^{-2} (F - \Psi)$$

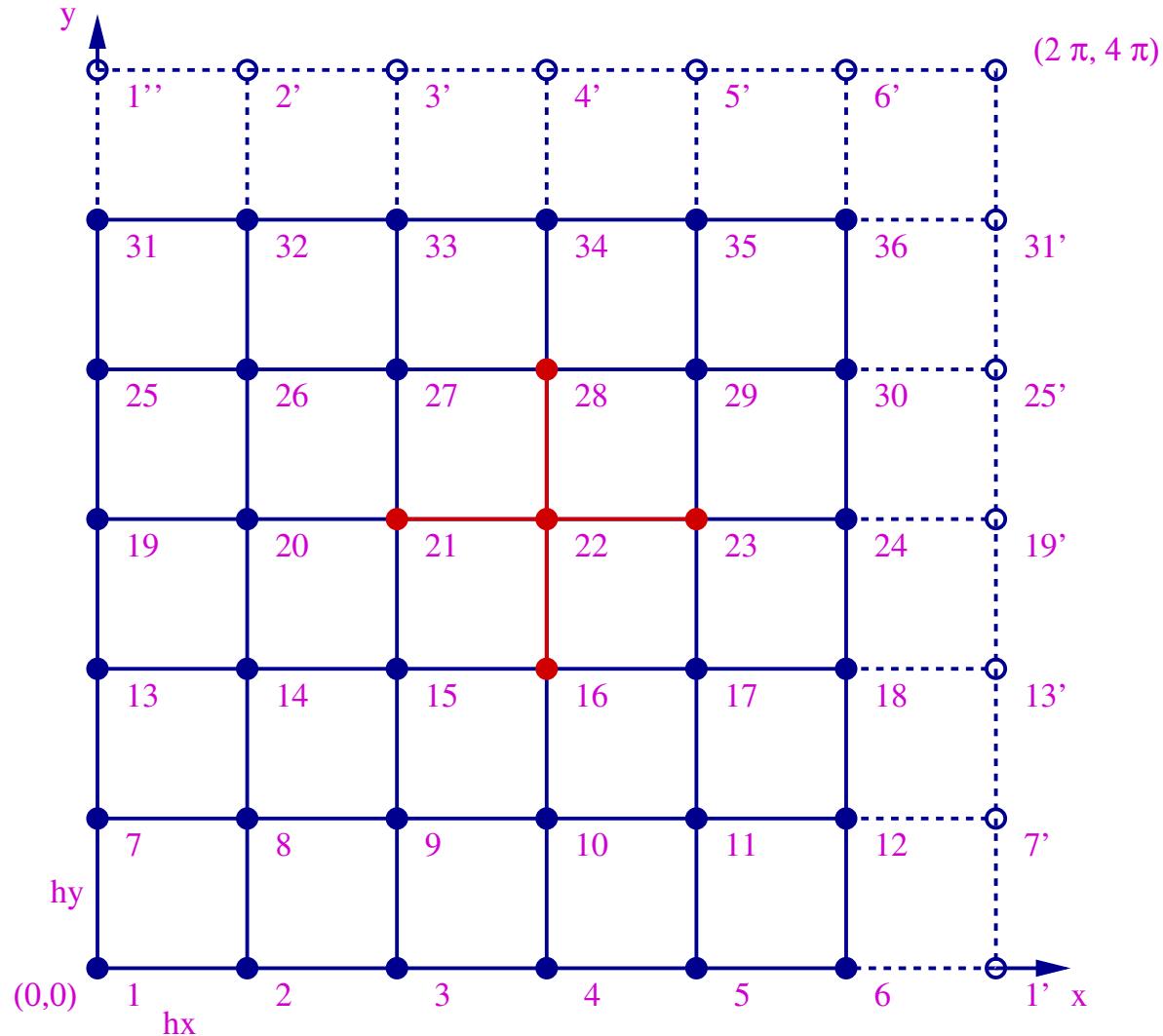
parameter values used in simulations

- μ (**viscosity**): $\{10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}\}$
- η (**resistivity**): $\{10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}\}$
- d_e (**inertial skin depth**): $\{0.08\}$
- ρ_s (**ion sound larmor radius**): $\{0.08, 0.16, 0.24, 0.32, 0.40, 0.48, 0.56, 0.64, 0.72, 0.80\}$

space and time discretizations

- standard, five point stencil, second order accurate space discretization.
- up to second order accurate time discretization for the explicit code, based on adams' formulas.
- up to fourth order accurate time discretization for the implicit code, based on backward differentiation formulas (bdf).

space discretization (rectangular grid)



discrete system for explicit solution (first order time accurate)

$$\frac{\Omega_{n+1} - \Omega_n}{\Delta t} + [\Phi_n, \Omega_n] = d_e^{-2} [F_n, \Psi_n] + \mu \nabla^2 \Omega_n$$

$$\frac{F_{n+1} - F_n}{\Delta t} + [\Phi_n, F_n] = \rho_s^2 [\Omega_n, \Psi_n] + \eta \nabla^2 (\Psi_n - \Psi^e)$$

$$\nabla^2 \Phi_{n+1} = \Omega_{n+1}$$

$$\nabla^2 \Psi_{n+1} = d_e^{-2} (\Psi_{n+1} - F_{n+1})$$

discrete system for implicit solution (first order time accurate)

$$\frac{\Omega_{n+1} - \Omega_n}{\Delta t} + [\Phi_{n+1}, \Omega_{n+1}] = d_e^{-2}[F_{n+1}, \Psi_{n+1}] + \mu \nabla^2 \Omega_{n+1}$$

$$\frac{F_{n+1} - F_n}{\Delta t} + [\Phi_{n+1}, F_{n+1}] = \rho_s^2[\Omega_{n+1}, \Psi_{n+1}] + \eta \nabla^2(\Psi_{n+1} - \Psi^e)$$

$$\nabla^2 \Phi_{n+1} = \Omega_{n+1}$$

$$\nabla^2 \Psi_{n+1} = d_e^{-2}(\Psi_{n+1} - F_{n+1})$$

explicit time discretizations (for $u'(t) = f(t, u(t))$)

$$u_{n+1} = u_n + hf_n$$

$$u_{n+1} = u_n + \frac{3h}{2}f_n - \frac{h}{2}f_{n-1}$$

$$u_{n+1} = u_n + \left(h_b + \frac{h_b^2}{2h_a} \right) f_n - \frac{h_b^2}{2h_a} f_{n-1}$$

implicit time discretizations (for $u'(t) = f(t, u(t))$)

$$u_{n+1} = u_n + hf_{n+1}$$

$$u_{n+1} = \frac{4}{3}u_n - \frac{1}{3}u_{n-1} + \frac{2h}{3}f_{n+1}$$

$$u_{n+1} = \frac{18}{11}u_n - \frac{9}{11}u_{n-1} + \frac{2}{11}u_{n-2} + \frac{6h}{11}f_{n+1}$$

$$u_{n+1} = \frac{48}{25}u_n - \frac{36}{25}u_{n-1} + \frac{16}{25}u_{n-2} - \frac{3}{25}u_{n-3} + \frac{12h}{25}f_{n+1}$$

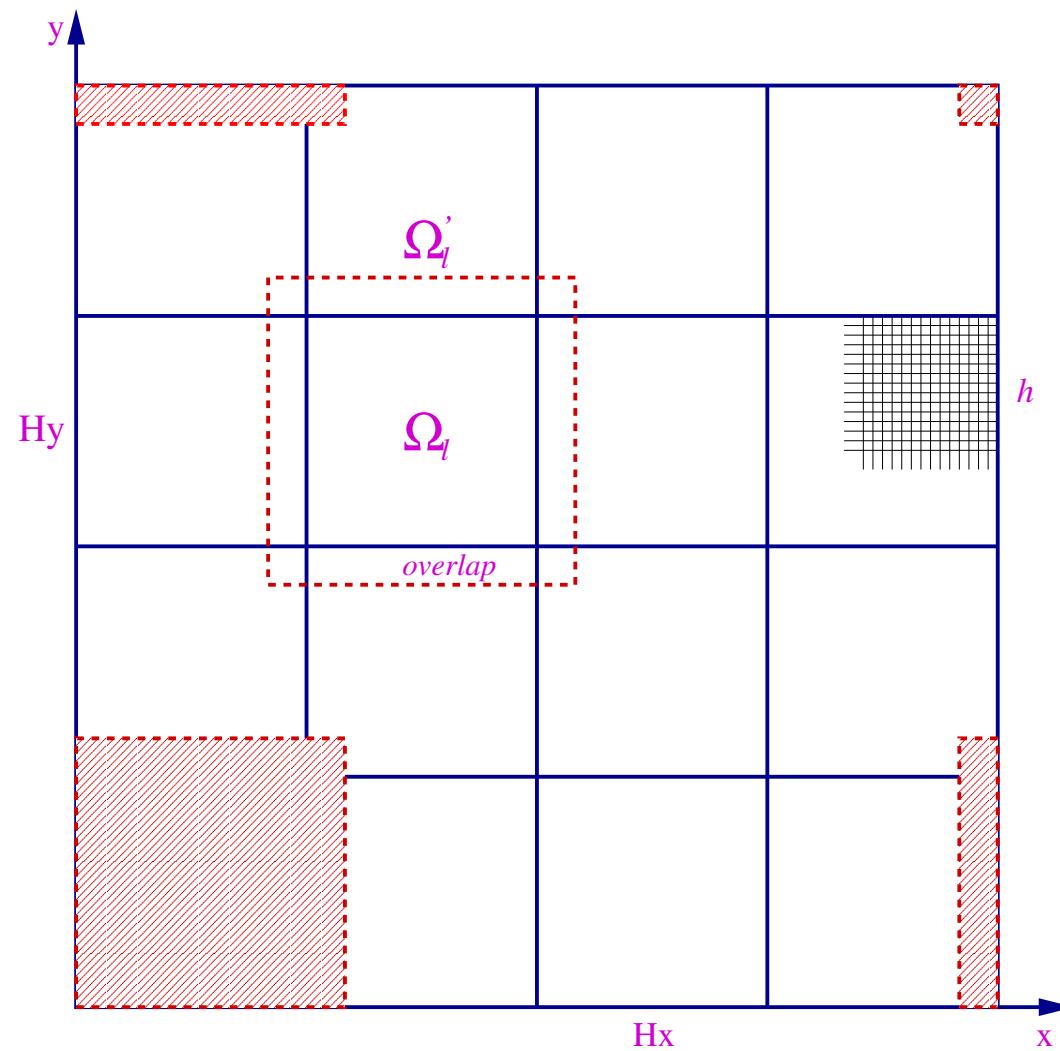
the newton-krylov-schwarz (nks) approach (1/2)

- at each time step we solve a nonlinear system $\mathcal{F}(u) = 0$, where $u = \{\Omega, F, \Phi, \Psi\}$.
- the nonlinear system is solved with a line search based newton iteration: $u_{k+1} = u_k - \lambda_k J^{-1}(u_k) \mathcal{F}(u_k)$ (where $k = 0, 1, \dots$).
- u_0 is the solution from the previous time step (initial condition for the first time step) and λ_k is the step length determined by the line search procedure.
- $J(u_k) = \mathcal{F}'(u_k)$ is the jacobian at u_k , singular because of the double periodic boundary conditions ($\dim(\text{null}(J(u_k))) = 1$ in this case).
- the jacobian is computed analytically, therefore there are no truncation errors in its computation.

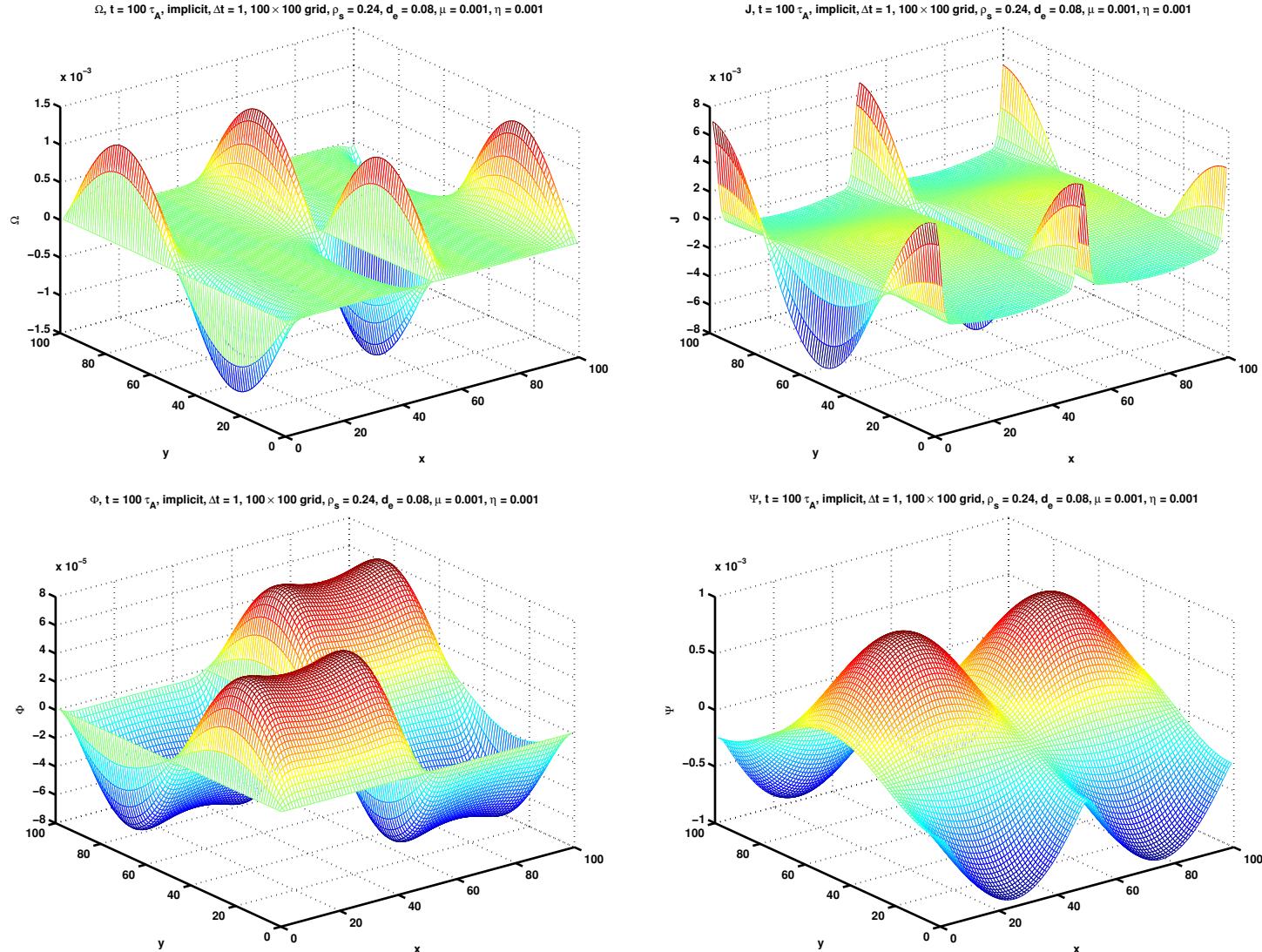
the newton-krylov-schwarz (nks) approach (2/2)

- the linear system $\Delta u_k = -\lambda_k J^{-1}(u_k)\mathcal{F}(u_k)$ is solved with a schwarz preconditioned krylov iteration.
- currently we employ a one level additive schwarz preconditioner and restarted gmres as the krylov accelerator.
- our convergence criteria are: relative nonlinear tolerance of 10^{-7} , absolute nonlinear tolerance of 10^{-7} , relative linear tolerance of 10^{-10} , absolute linear tolerance of 10^{-8} .
- for subdomains we use lu solves.

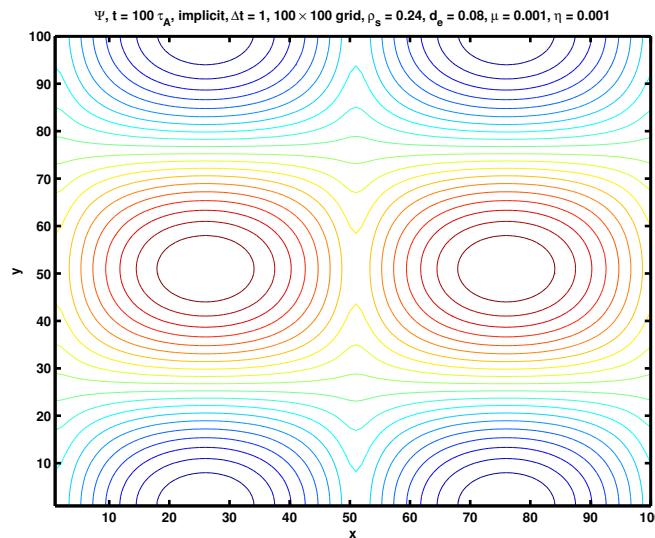
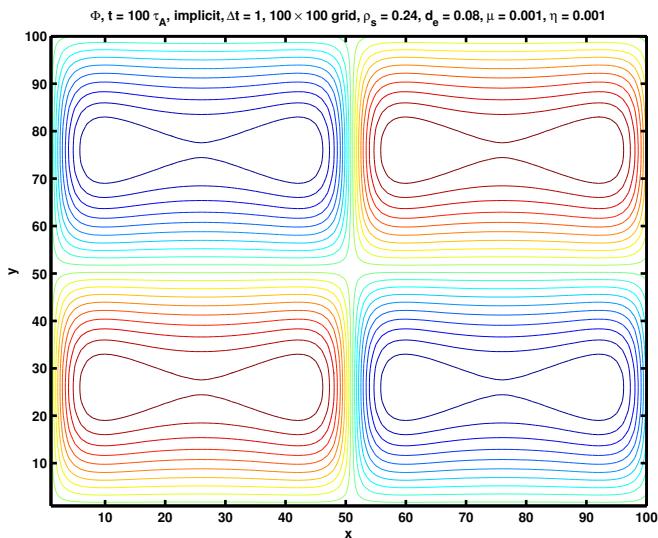
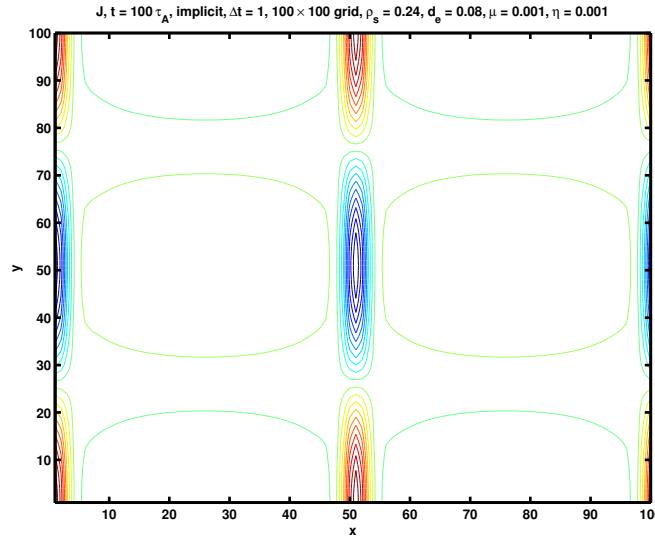
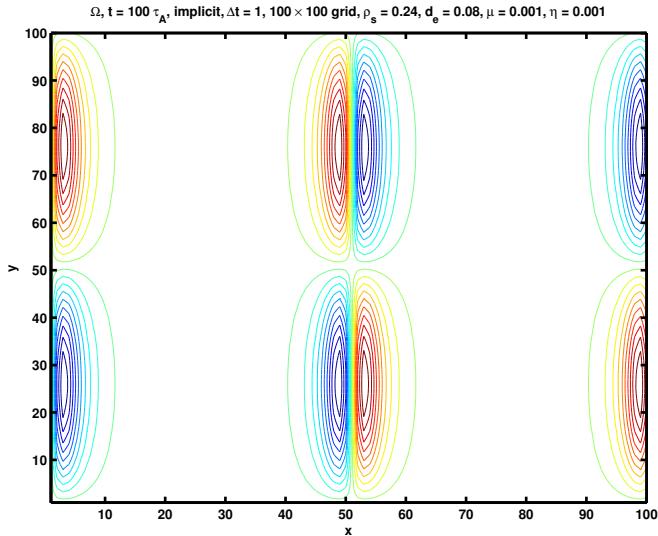
domain decomposition



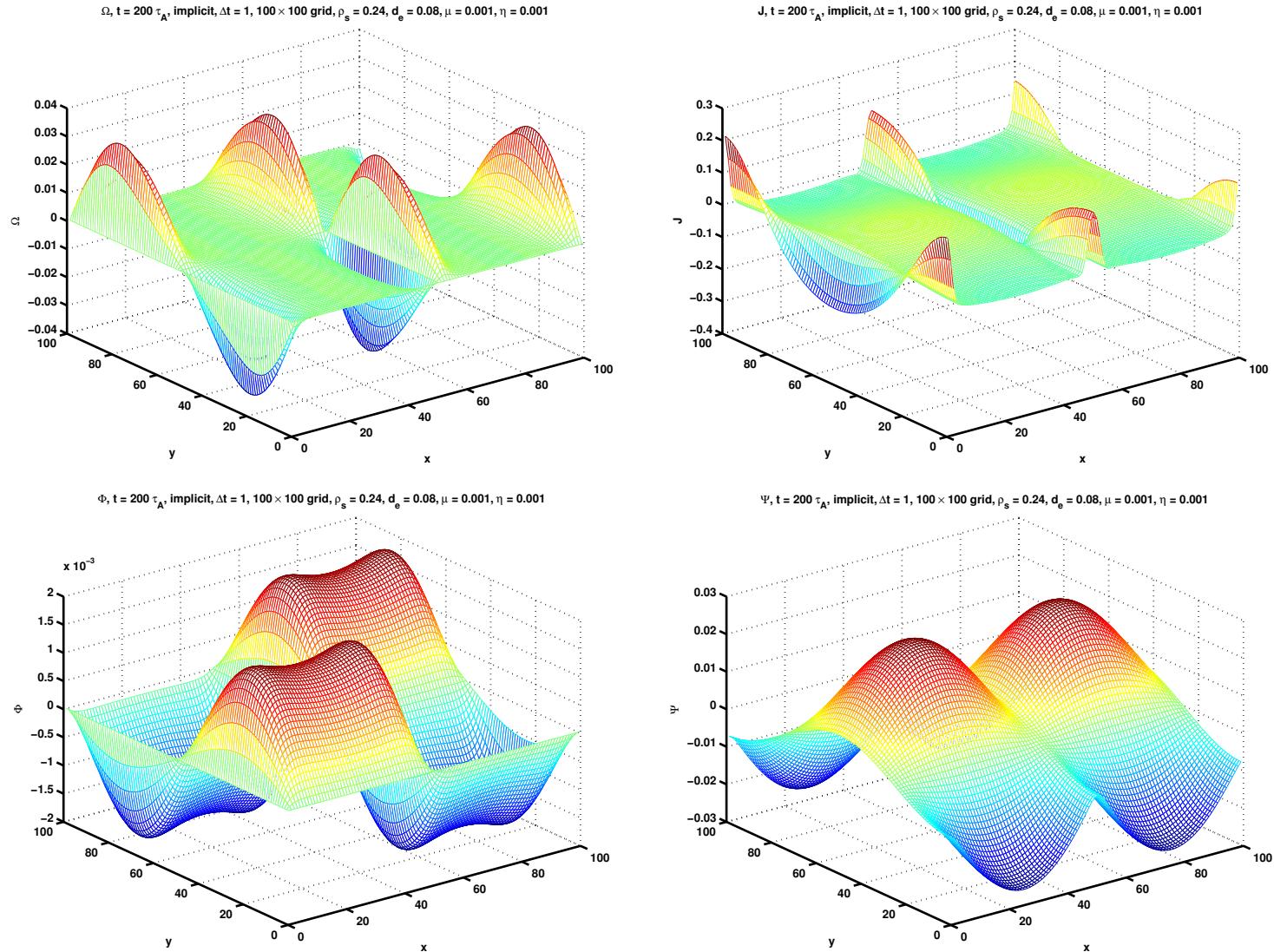
solution at $t = 100\tau_A$ (1/2)



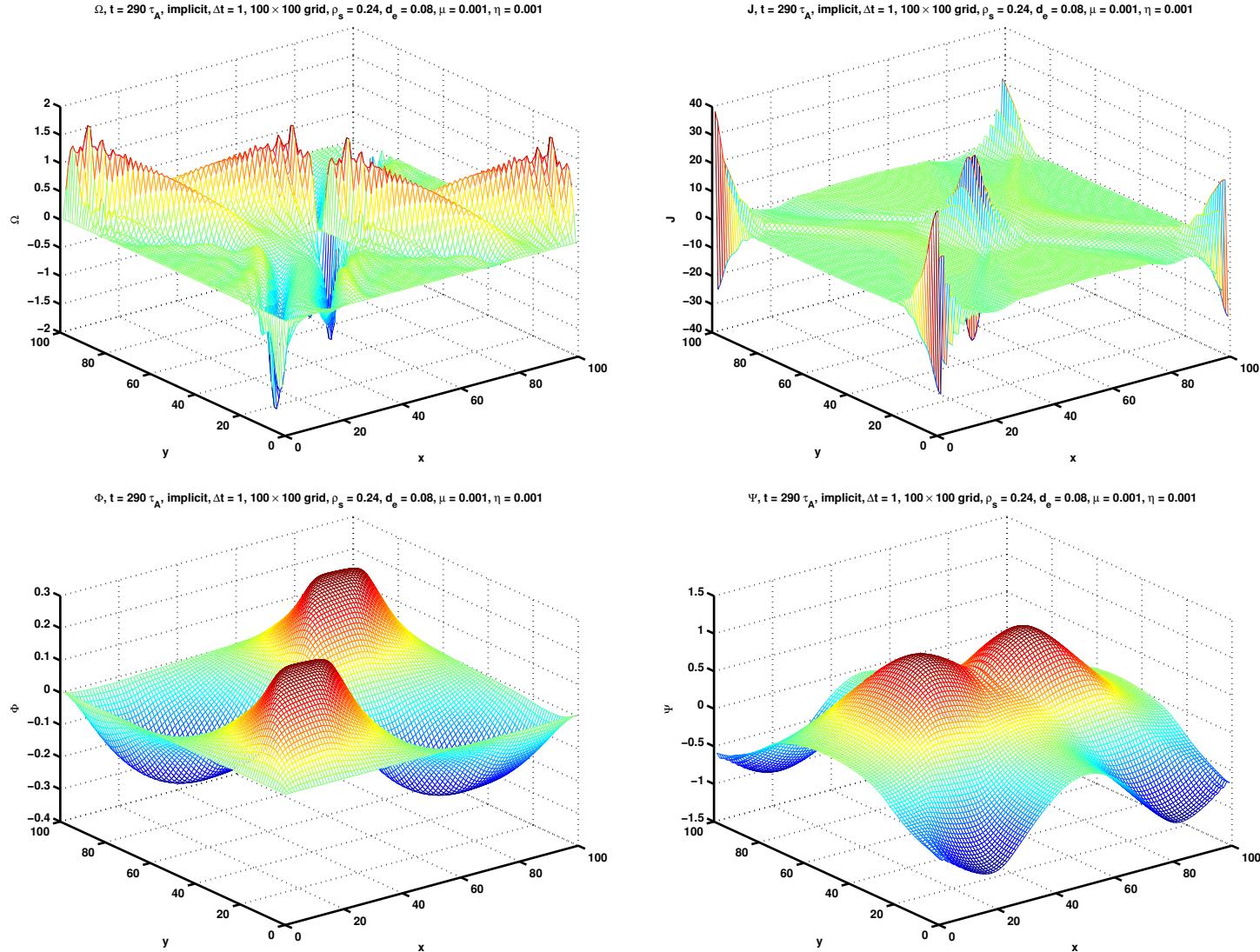
solution at $t = 100\tau_A$ (2/2)



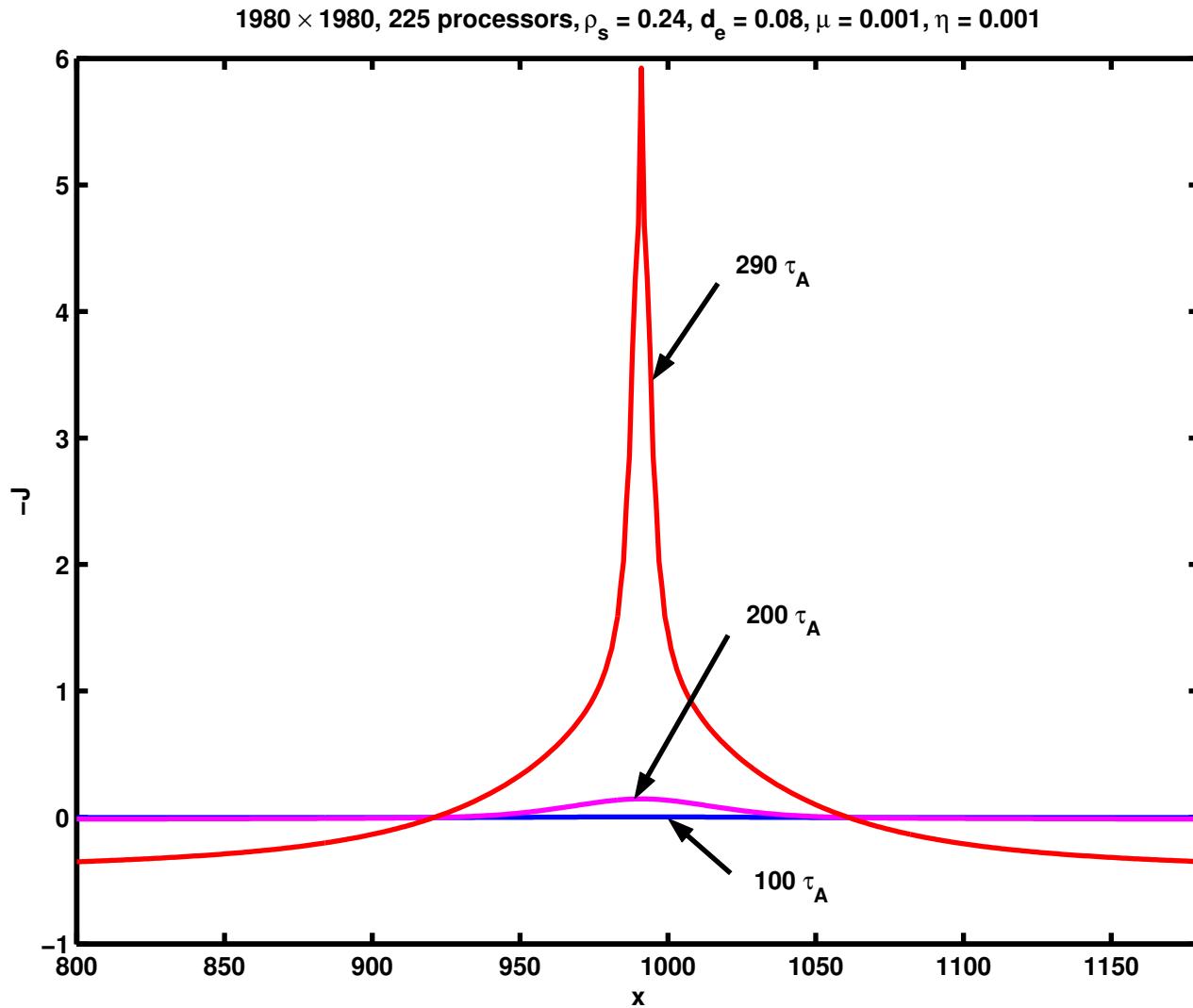
solution at $t = 200\tau_A$



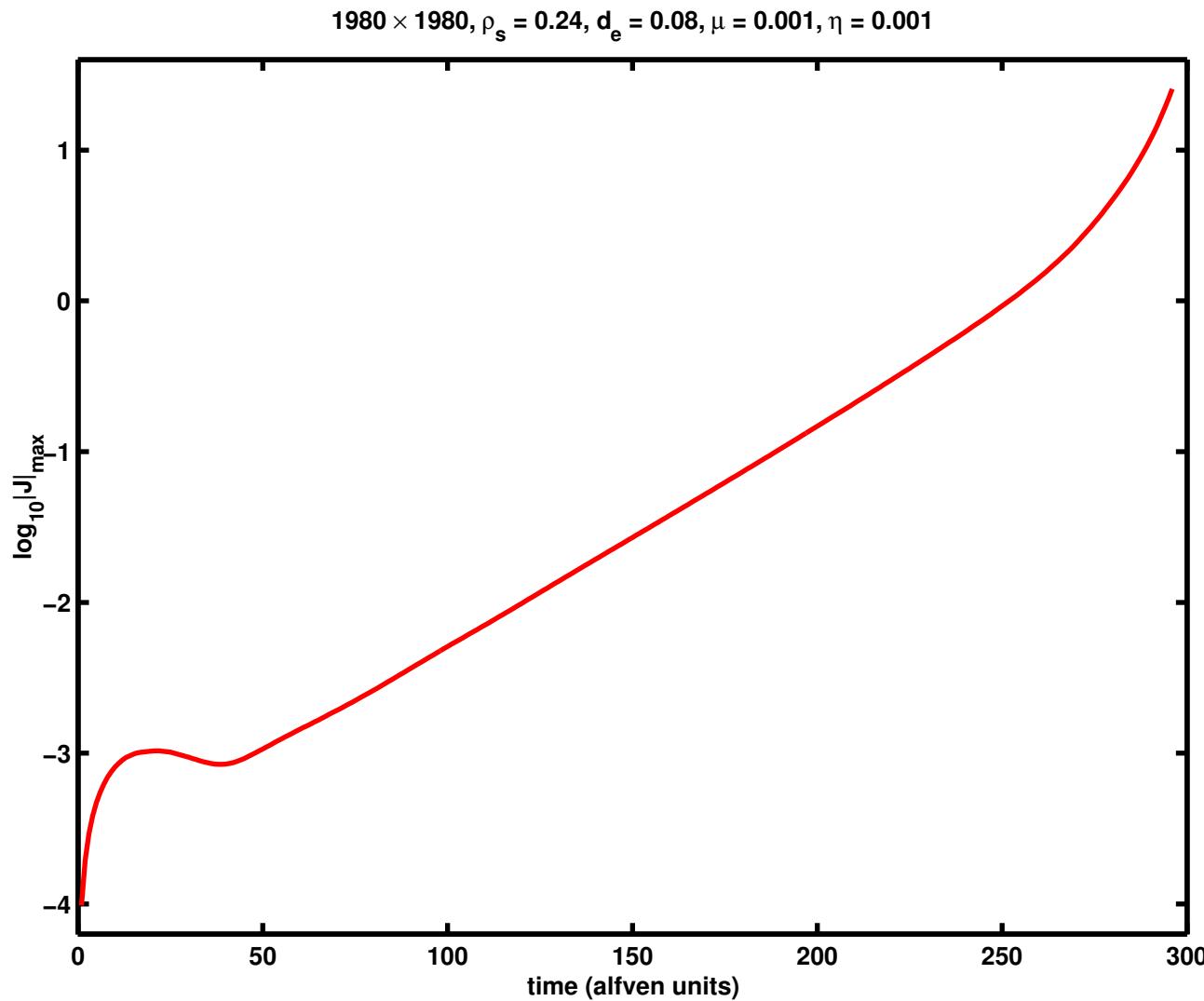
solution at $t = 290\tau_A$



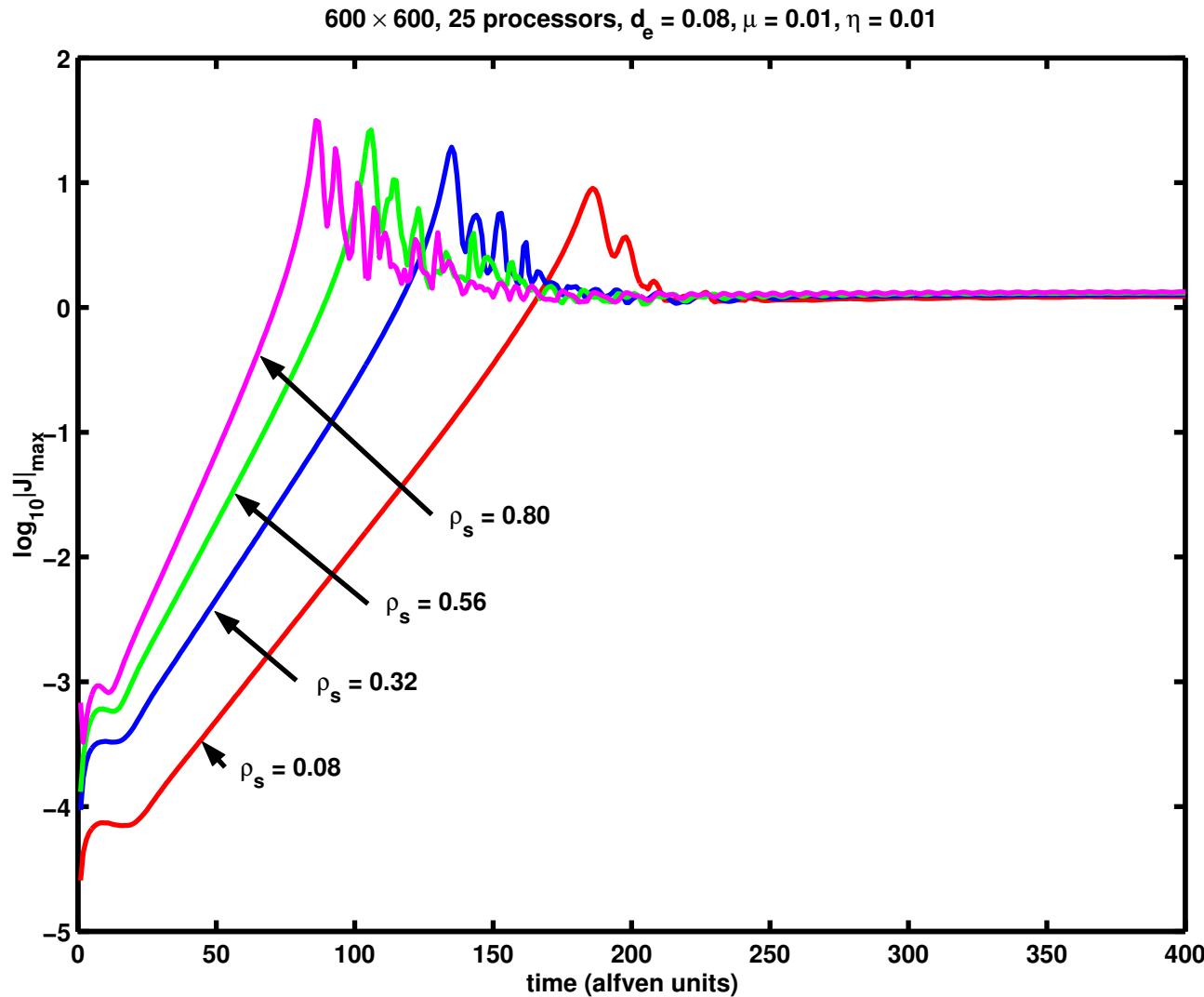
current density peaks



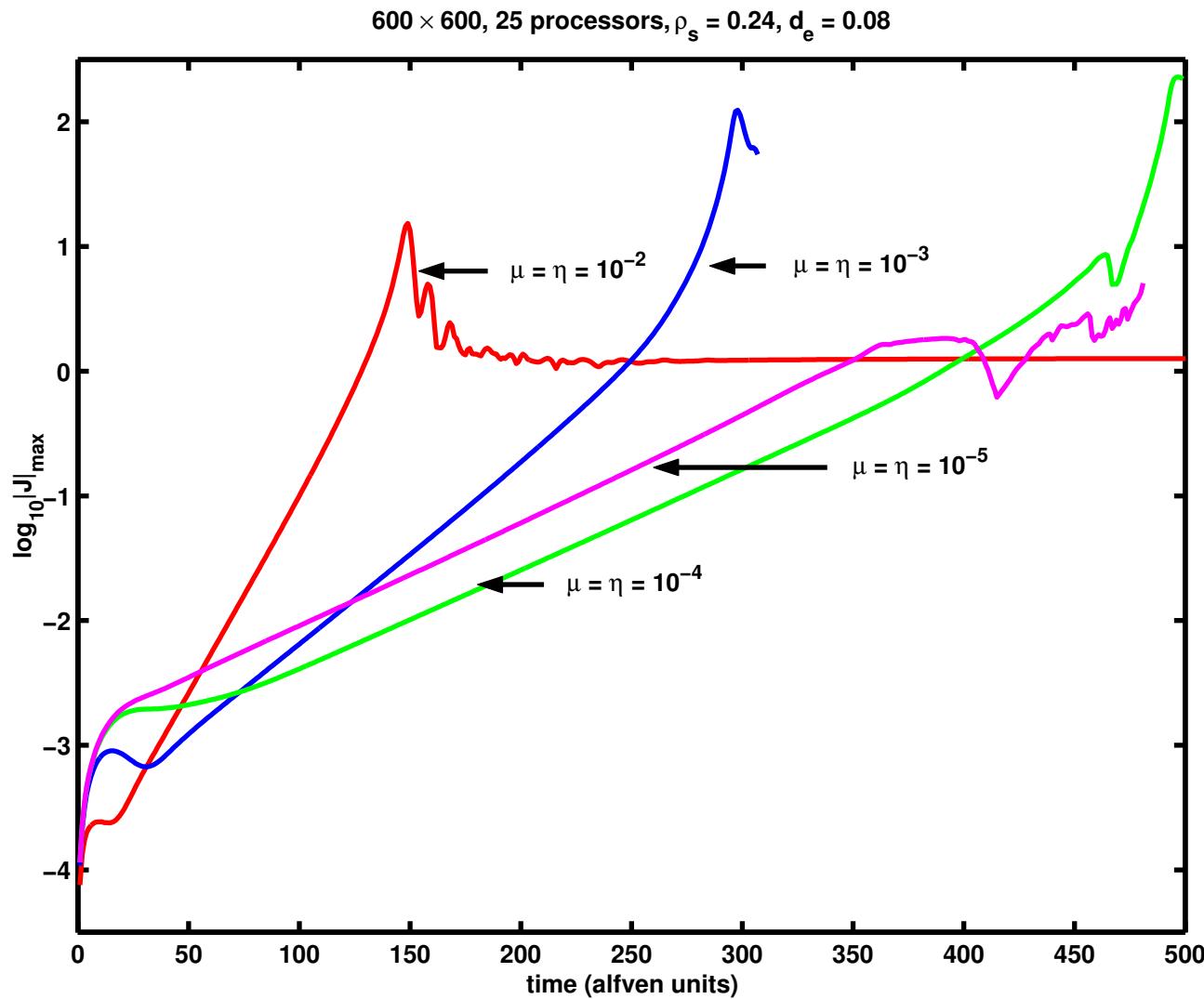
evolution of maximum current density



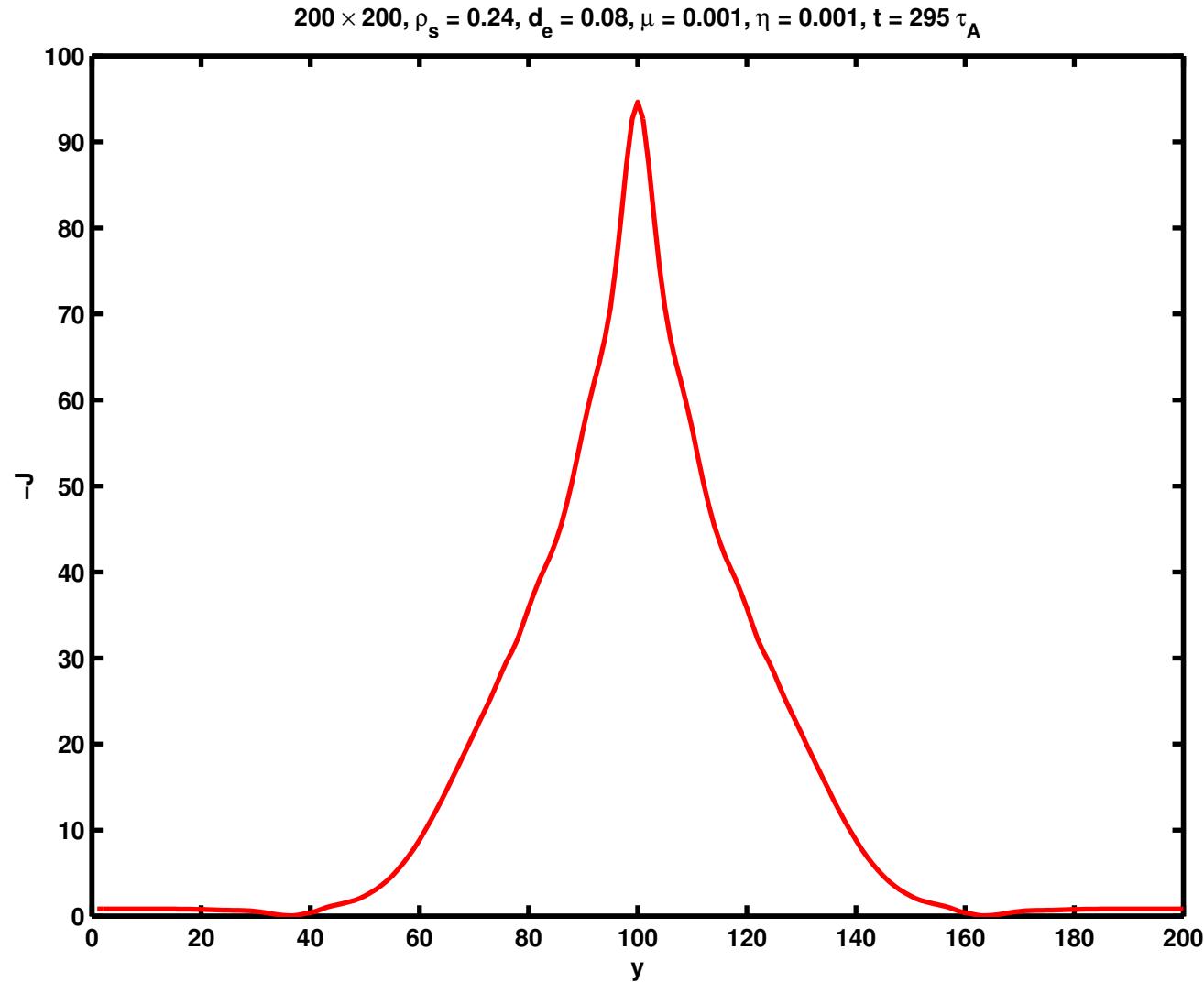
effect of the ρ_s/d_e ratio



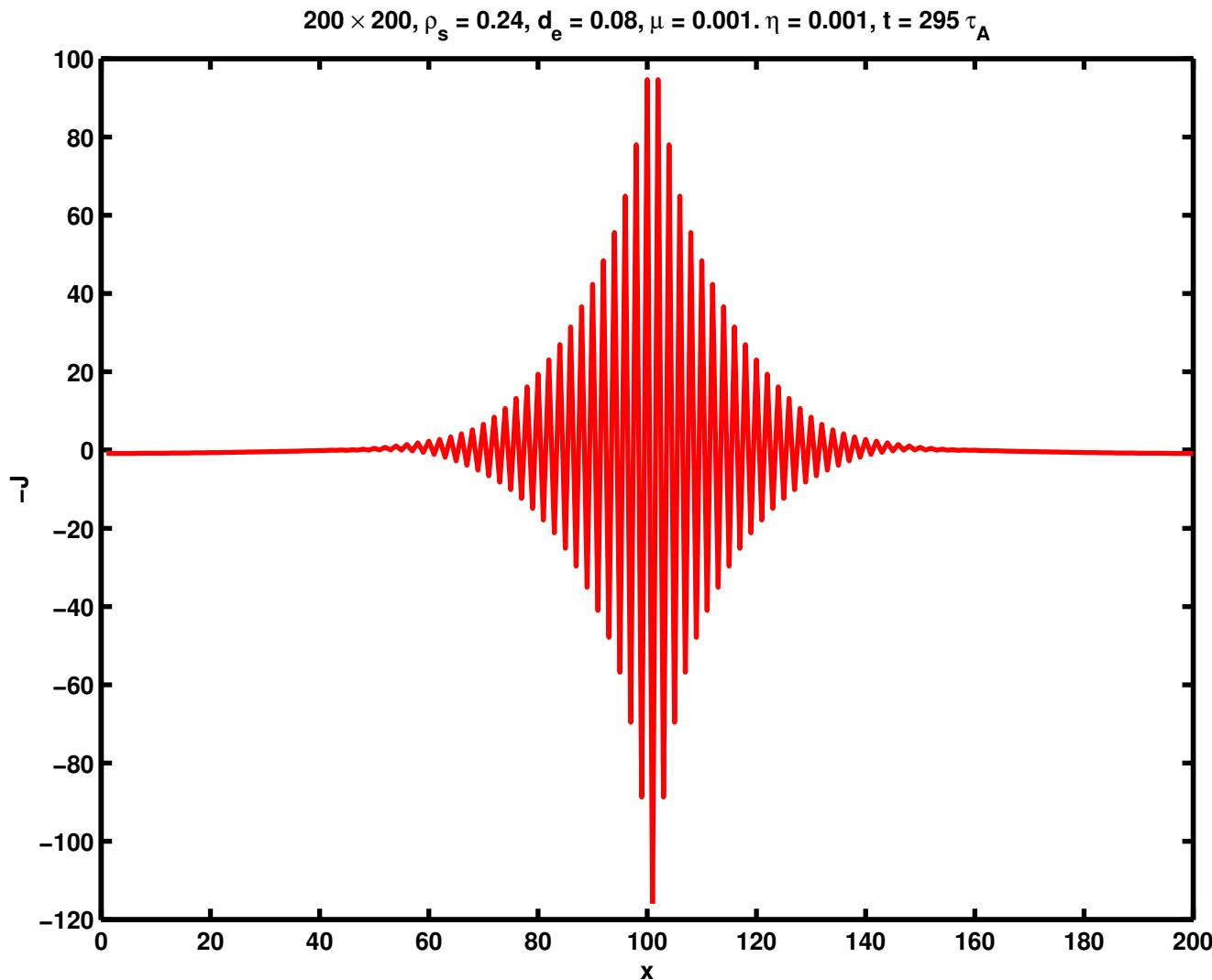
effect of the dissipation



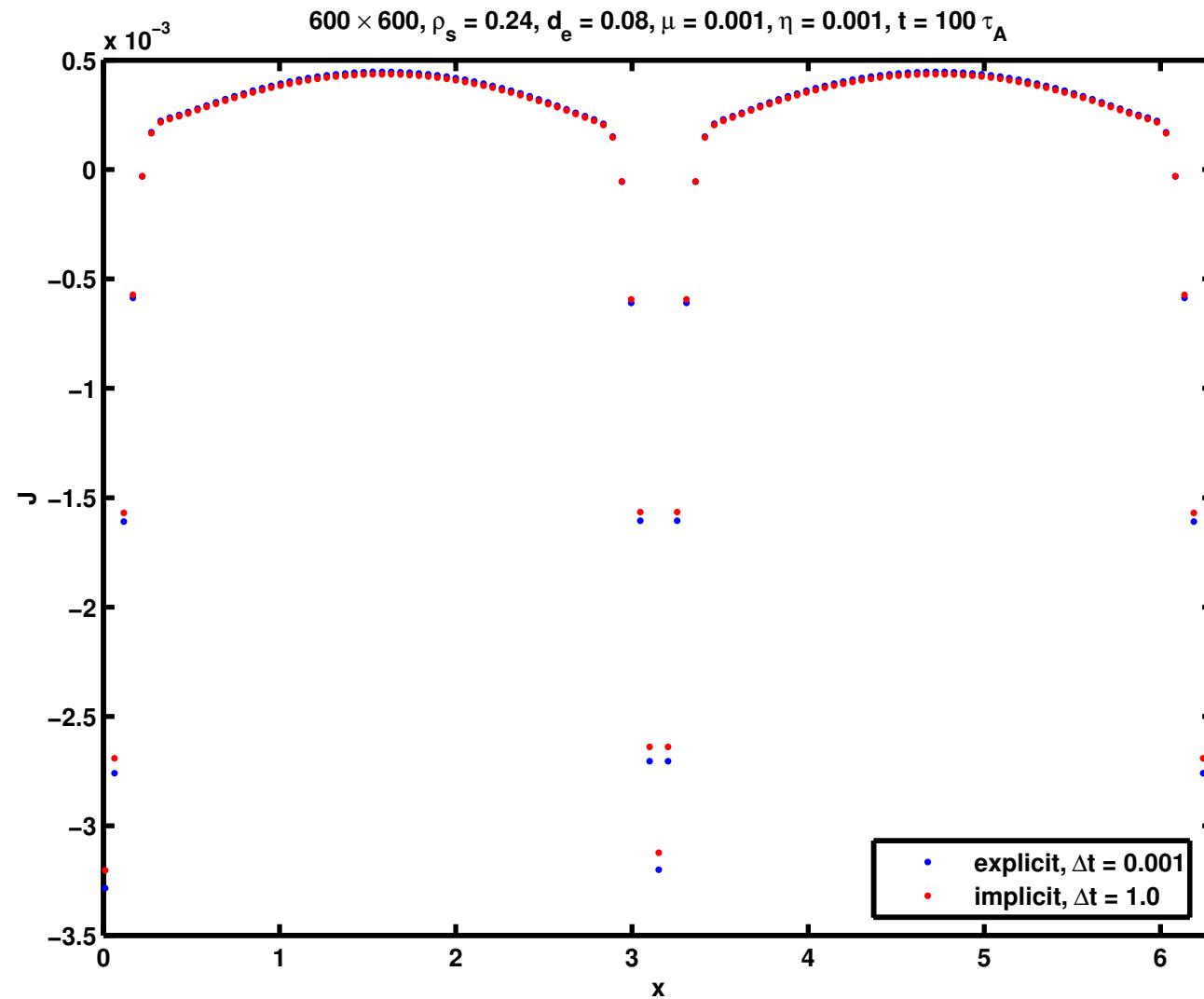
current density profiles in the nonlinear phase (1/2)



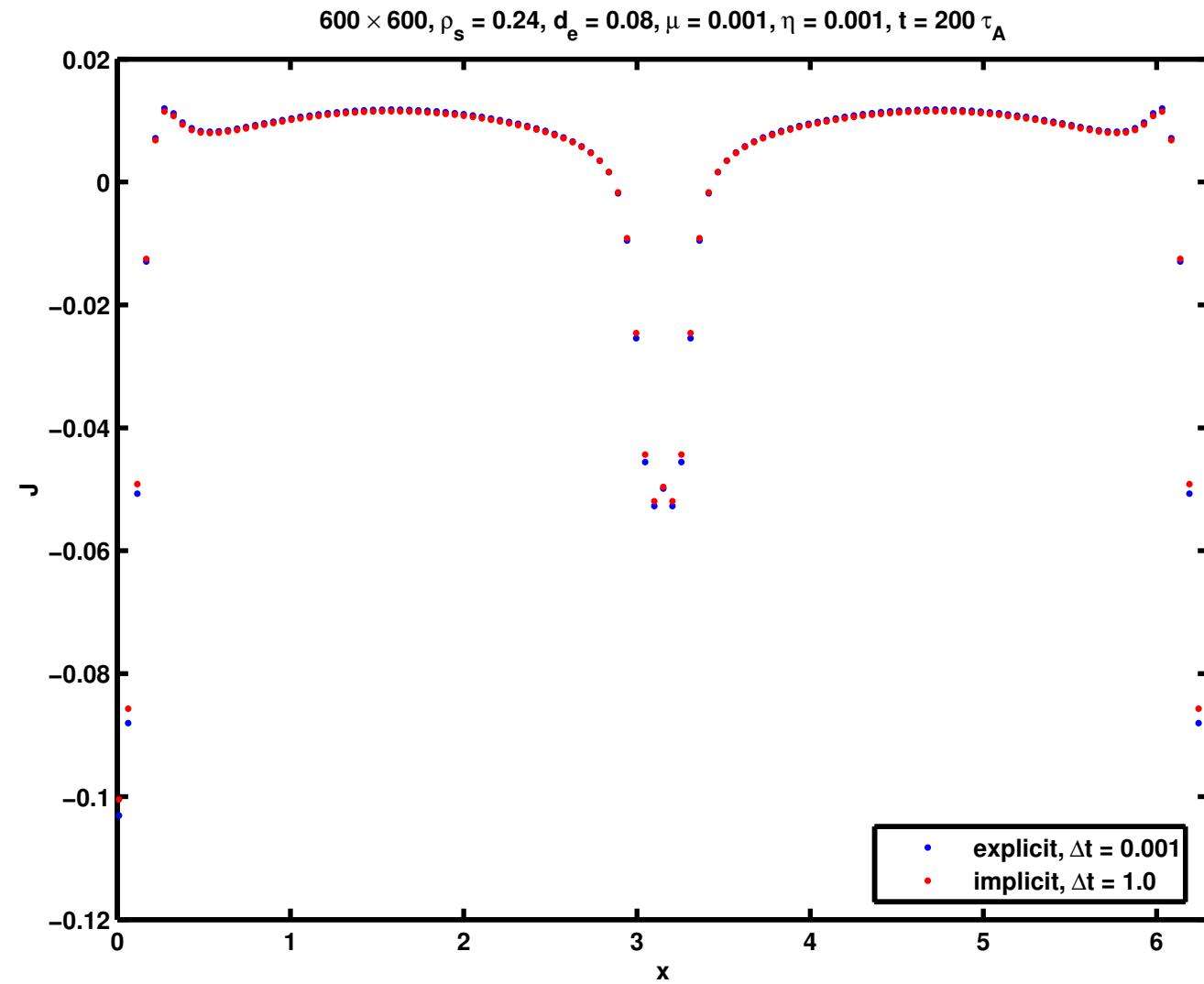
current density profiles in the nonlinear phase (2/2)



explicit/implicit current density profile comparison ($t = 100\tau_A$)



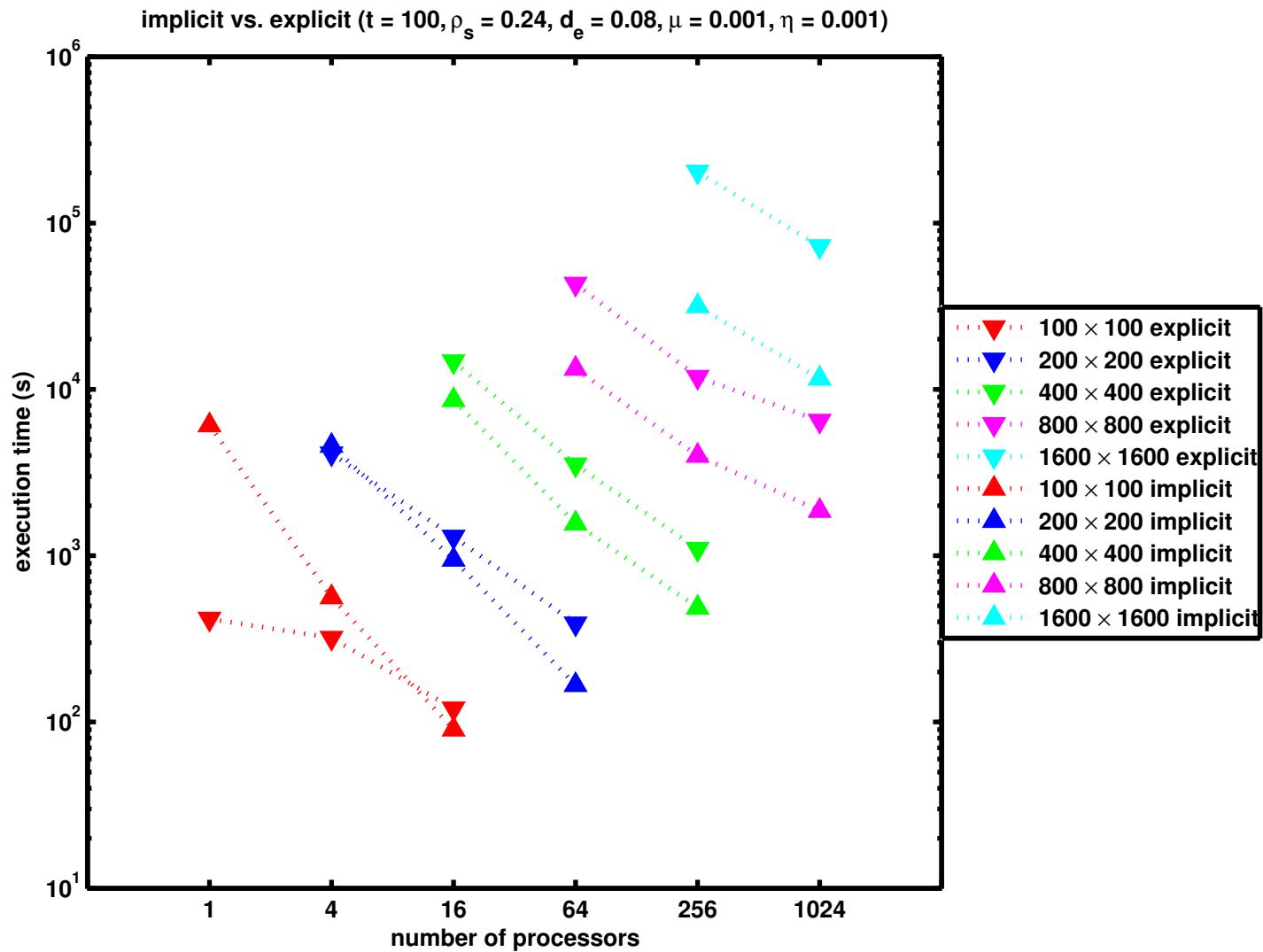
explicit/implicit current density profile comparison ($t = 200\tau_A$)



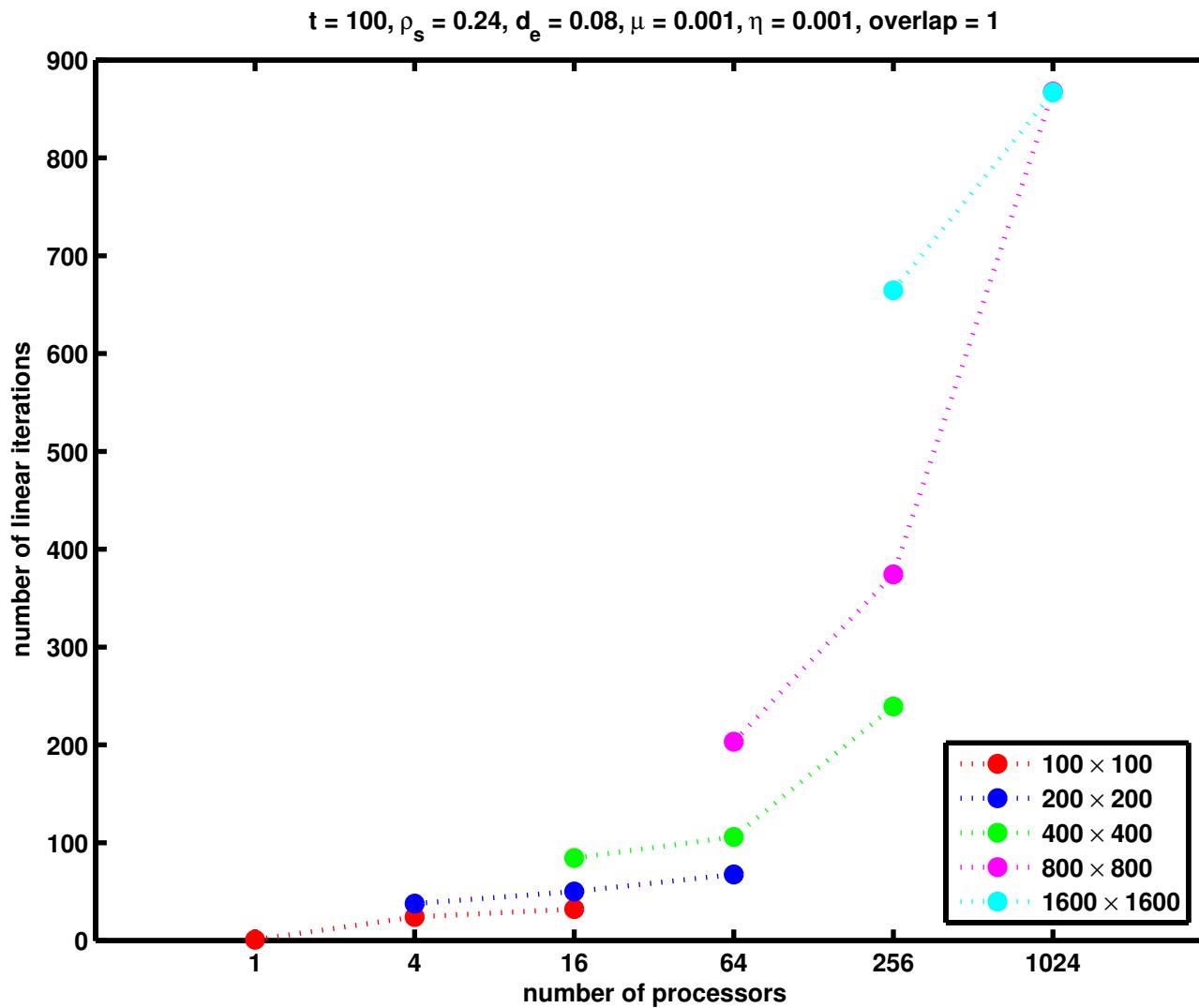
explicit/implicit time step size comparison ($t = 100\tau_A$)

grid	explicit Δt	implicit Δt
100×100	0.02	1
200×200	0.008	1
400×400	0.004	1
800×800	0.002	1
1600×1600	0.001	1

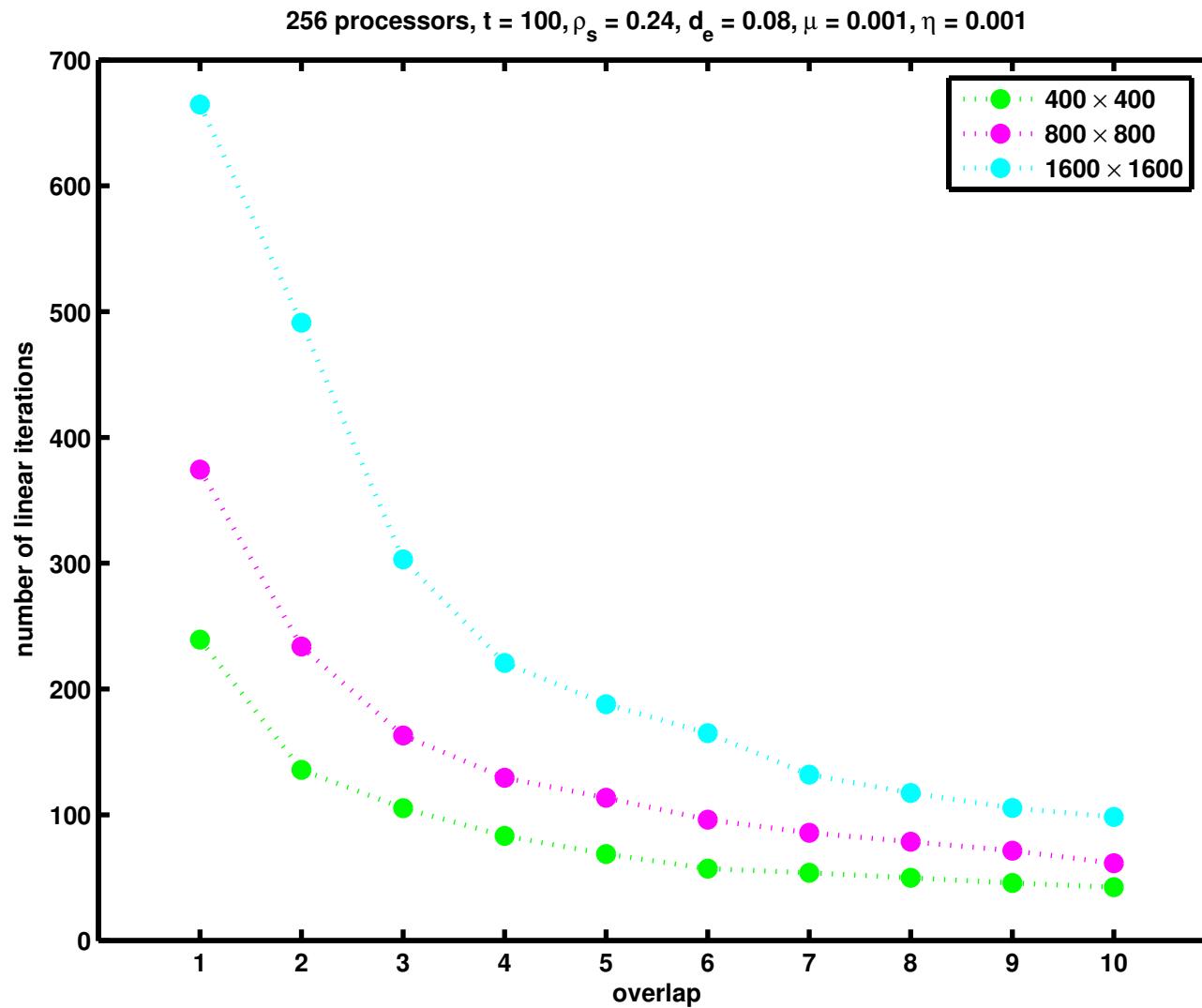
explicit/implicit execution time comparison



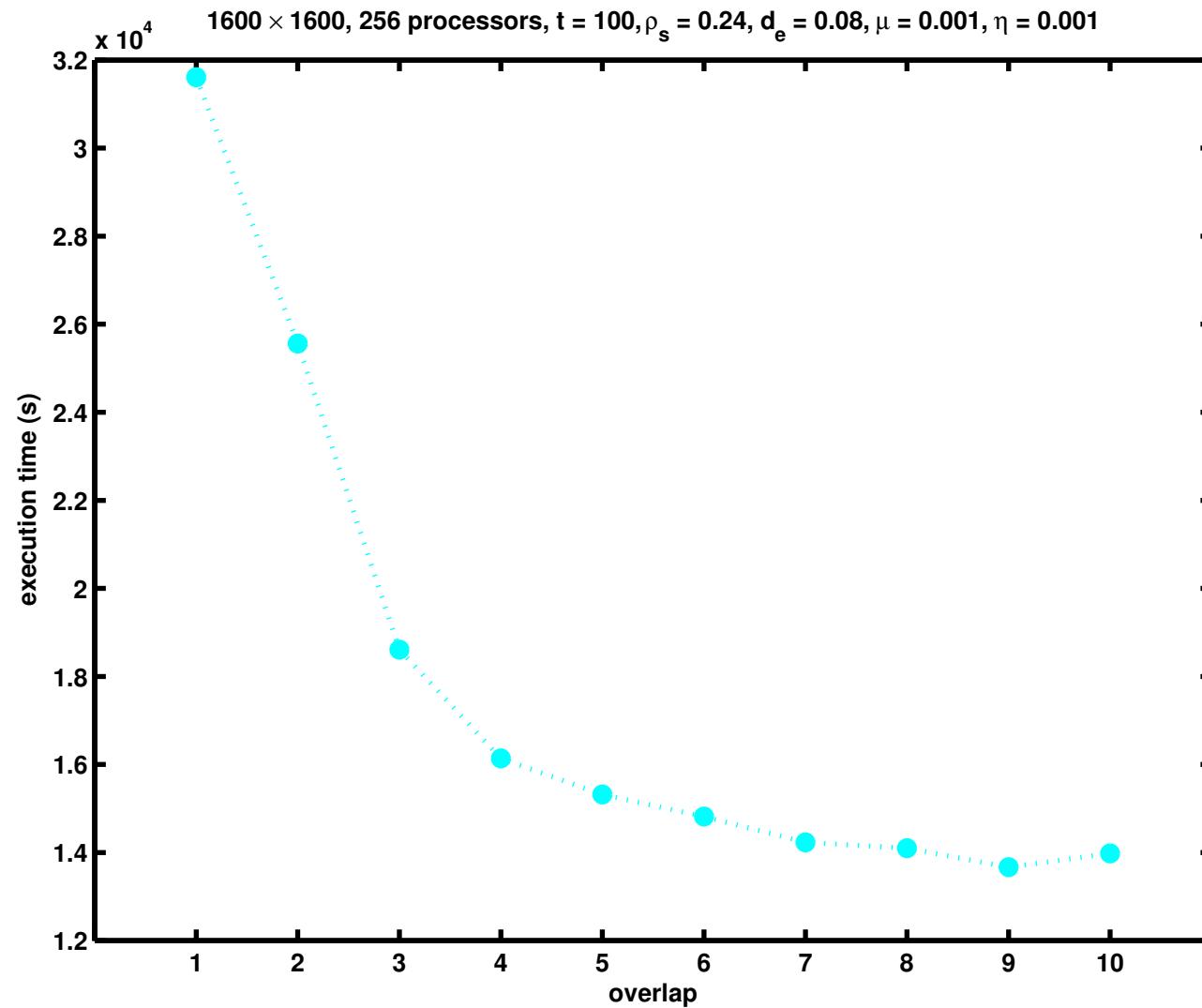
number of linear iterations (overlap = 1)



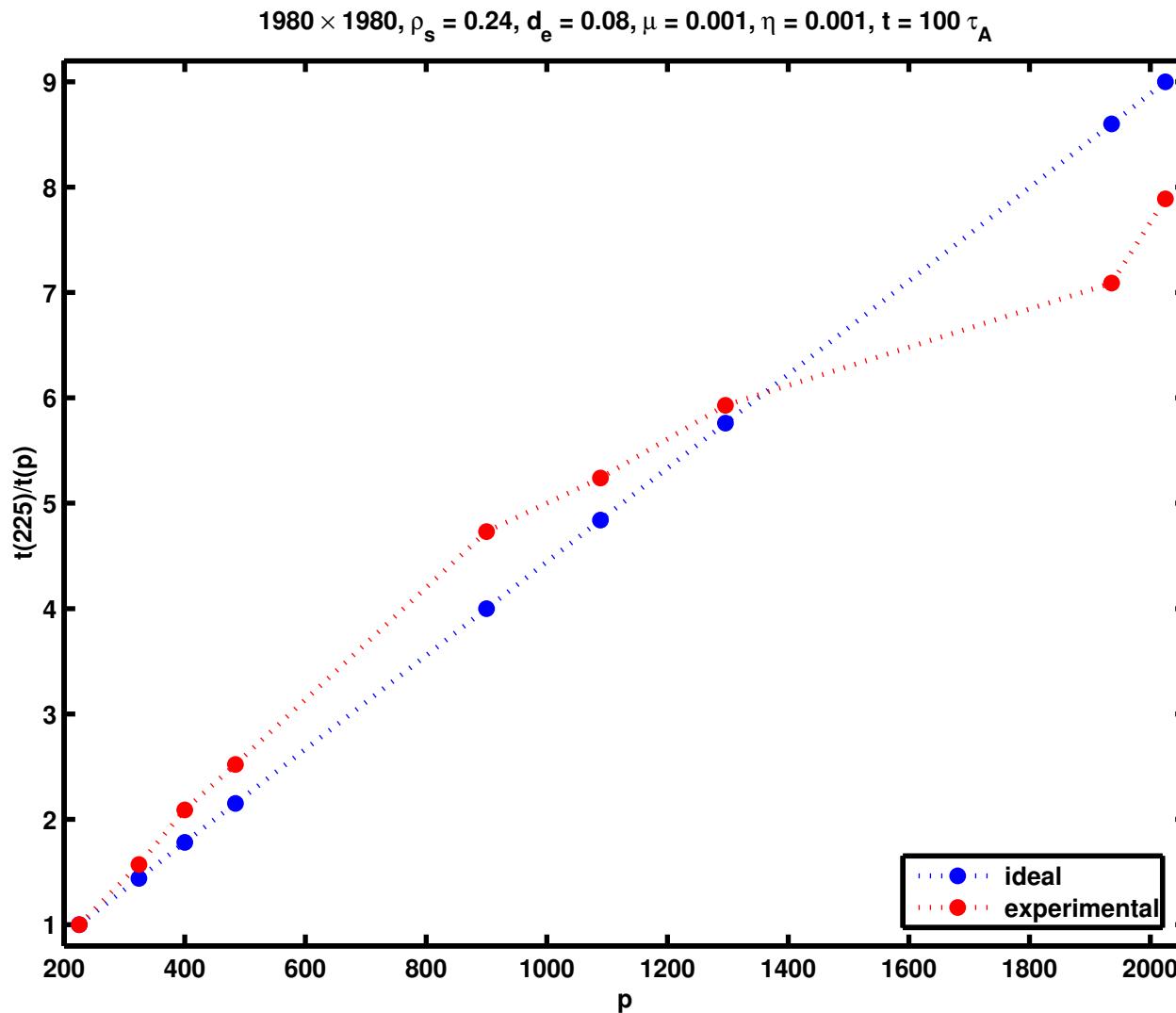
the effect of the overlap (1/2)



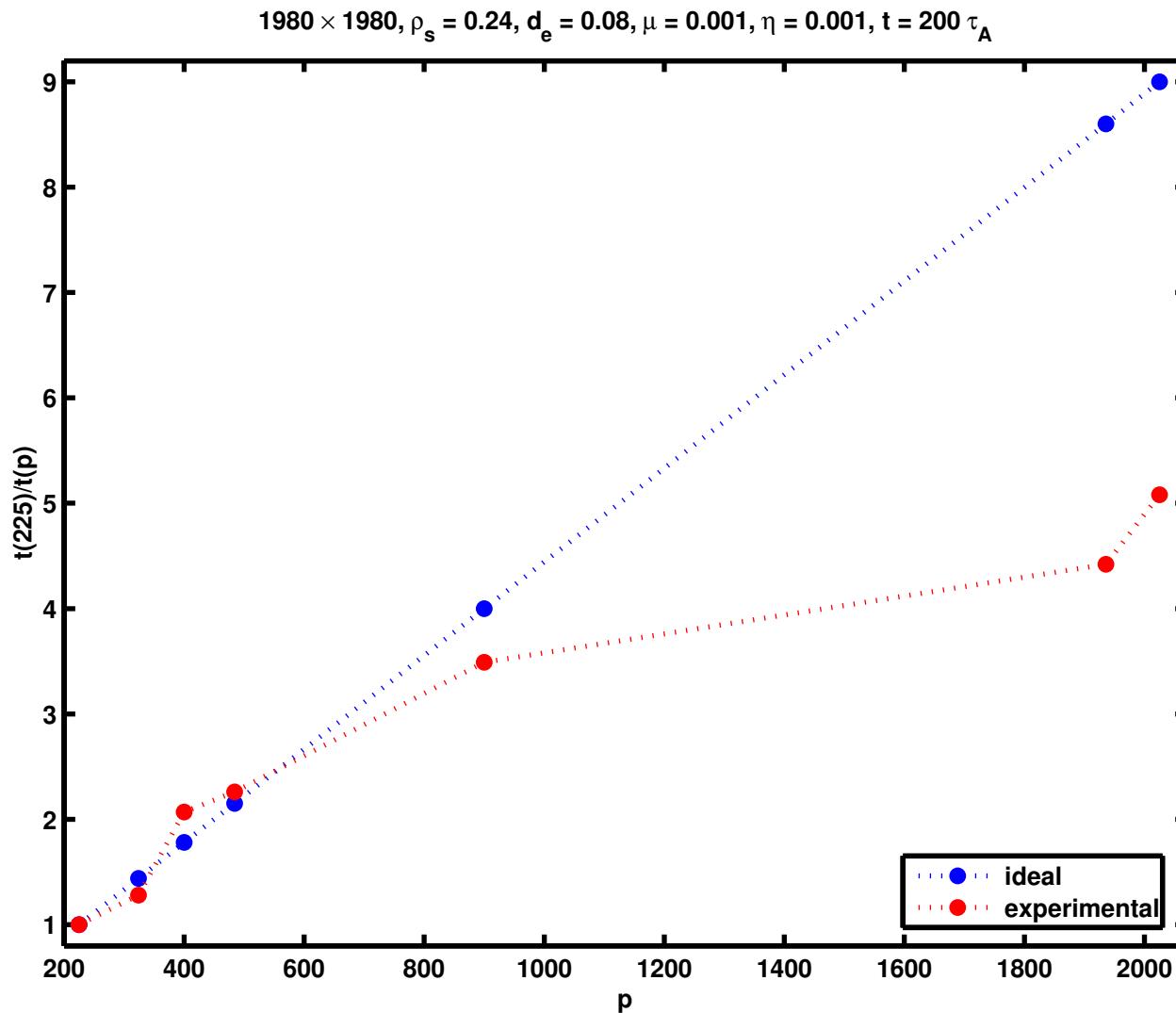
the effect of the overlap (2/2)



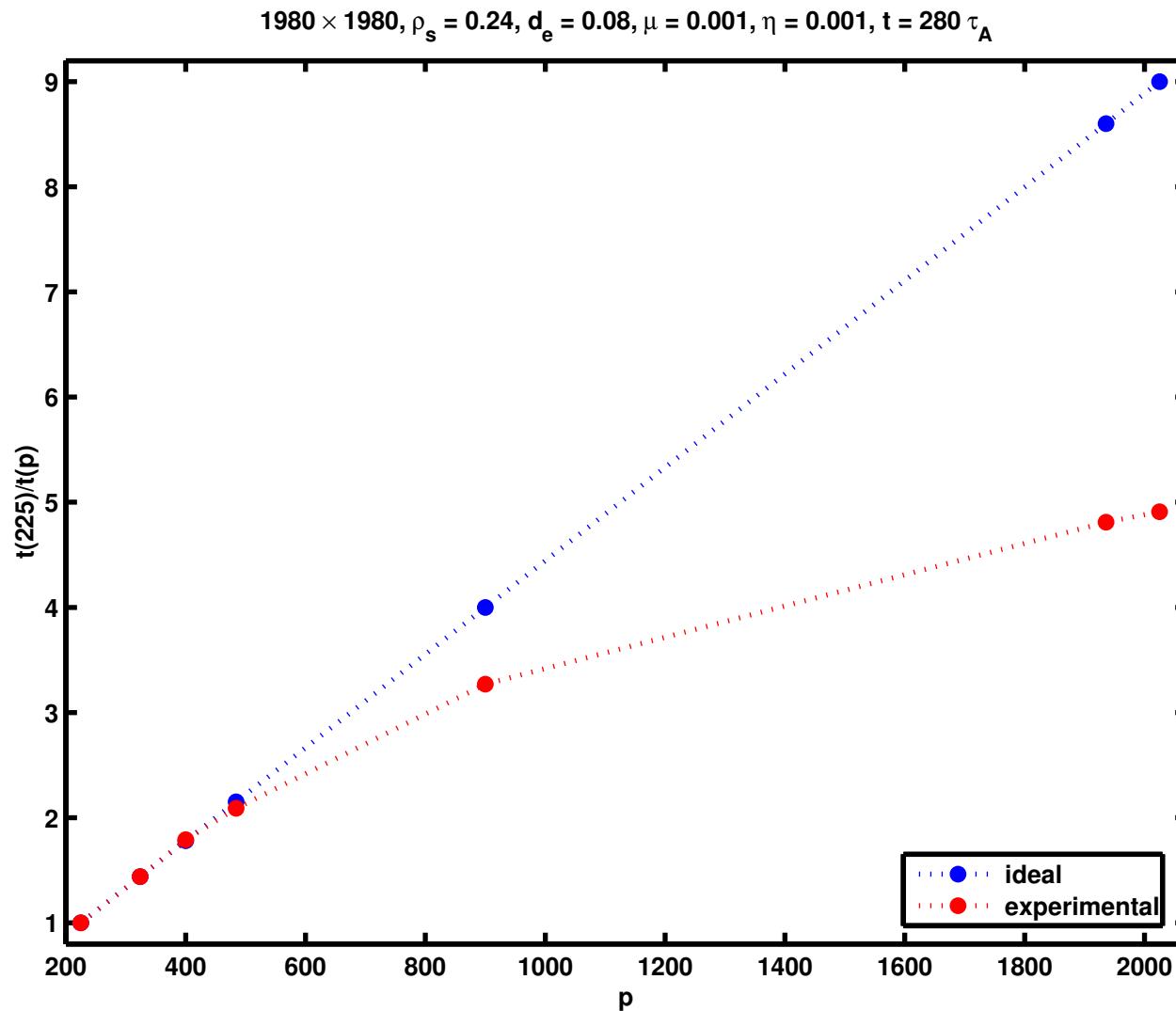
speedup at $t = 100\tau_A$



speedup at $t = 200\tau_A$



speedup at $t = 280\tau_A$



case study: tilt mode instability

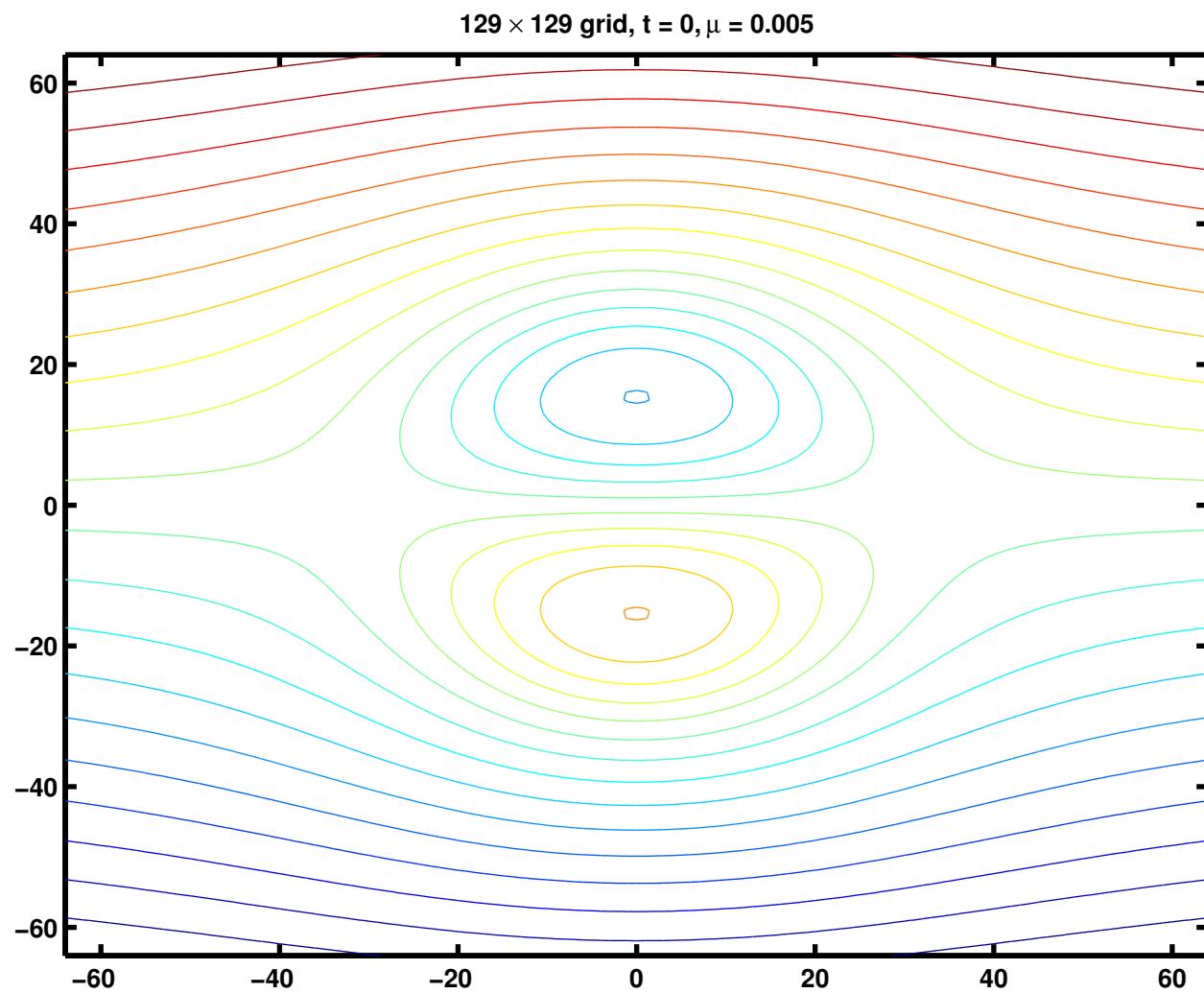
$$\frac{\partial \Omega}{\partial t} + [\Omega, \Phi] = [J, \Psi] + \mu \nabla^2 \Omega$$

$$\frac{\partial J}{\partial t} + [J, \Phi] = [\Omega, \Psi] + 2 \left[\frac{\partial \Phi}{\partial x}, \frac{\partial \Psi}{\partial x} \right] + 2 \left[\frac{\partial \Phi}{\partial y}, \frac{\partial \Psi}{\partial y} \right] + \eta \nabla^2 J$$

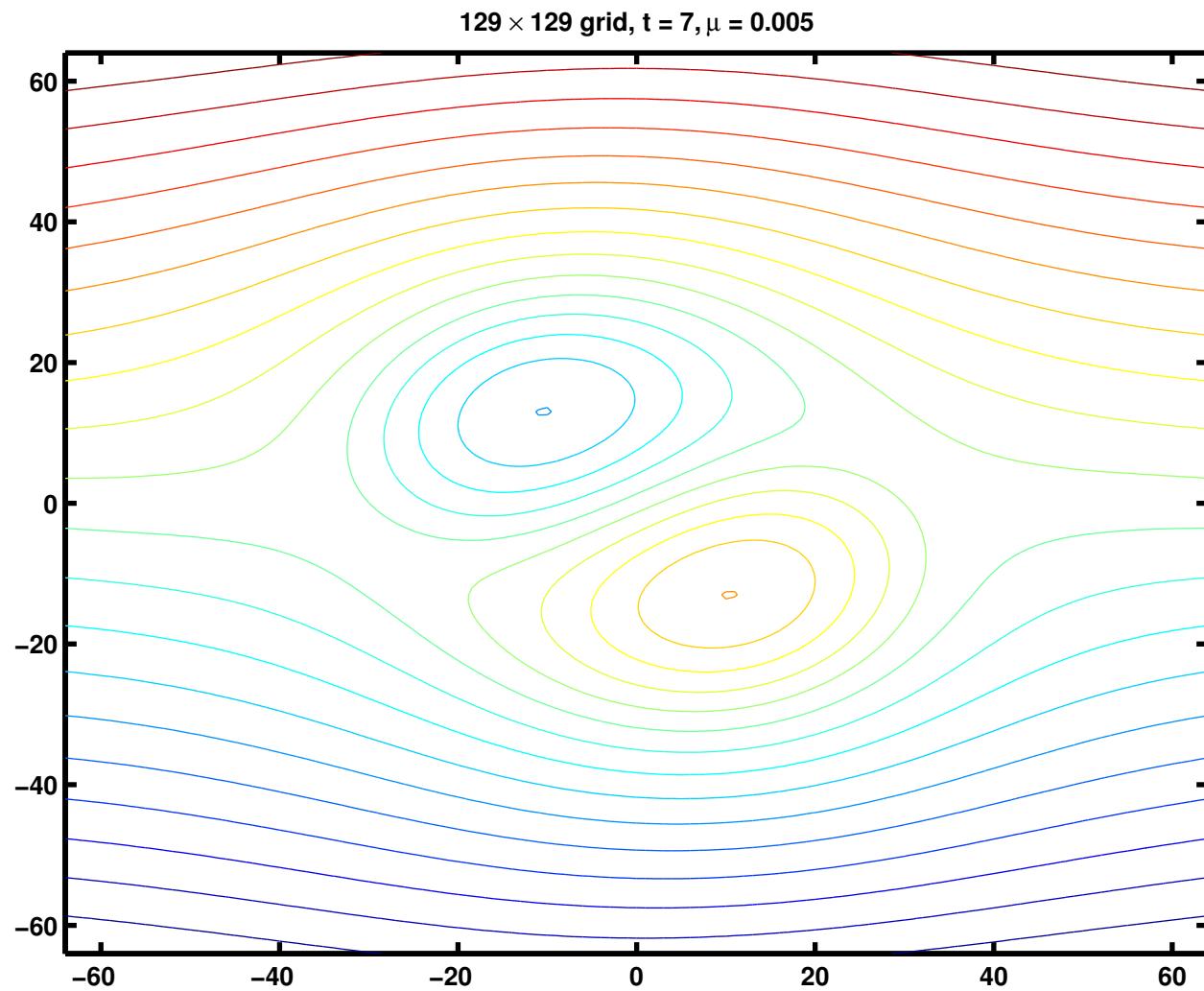
$$\nabla^2 \Phi = \Omega$$

$$\nabla^2 \Psi = J$$

solution at $t = 0\tau_A$



solution at $t = 7\tau_A$



conclusion

- we validate fully implicit nks solutions for model mhd problems.
- the implicit time steps are larger than the explicit time steps and this eventually leads to faster execution times for the implicit codes.
- several optimizations are possible for improved execution times.
- the full implicit nks solutions scale reasonably well, even with one level preconditioning.
- multilevel preconditioning would further reduce the number of iterations, which could determine even faster execution times.
- adaptivity is eventually required because of the multiscale nature of the problem.
- the highly nonlinear phase requires further investigation.
- employing the fully implicit approach in other mhd models could lead to a standard computational mhd test set.